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LETTER TO THE EDITOR

On 'conflict of conservation laws in cyclotron radiation'

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Abstract. We reconsider the apparent conflict of conservation laws in cyclotron radiation discovered by Lieu *et al* (1983). We show that they did not correctly include the effects of radiation reaction in their calculation. When a 'recoil' term, calculated using relativistic quantum theory, is included in the angular momentum of the particle the conflict disappears. We find that the guiding centre of the particle drifts outwards during cyclotron radiation.

Recently several letters have been published on the subject of an apparent conflict of conservation laws in cyclotron emission (Lieu *et al* 1983, hereafter referred to as LLE; Lieu *et al* 1984, Lieu 1984), and on the resolution of this conflict (DasGupta 1984). The conflict arose in comparing two calculations of $(d/dt)r_0^2$ (here r_0^2 is the square of the distance of the centre of gyration of a particle orbit from the z axis of the coordinate system, with the magnetic field parallel to the z axis), using conservation of angular momentum and conservation of linear momentum respectively. In this letter we reconsider the calculation which uses the law of conservation of angular momentum. LLE use it to find the angular momentum of the electron by first calculating the angular momentum of the radiation field. Here we calculate the angular momentum of the electron directly from its wavefunction. We discuss the emission process from two viewpoints: firstly, when a single measurement of the state of the system is made; and secondly, when the expectation value of the angular momentum is measured. In the latter case we include a 'recoil' term, calculated from relativistic quantum theory. This term was omitted by LLE: we show that it leads to a value for $(d/dt)r_0^2$ which agrees with that obtained using the law of conservation of linear momentum. This resolves the conflict. Our result differs from that given by DasGupta (1984). We believe that he erred in failing to carefully distinguish between the expectation value of a product and the product of expectation values.

The system consisting of a particle in a homogeneous magnetic field, in the absence of radiation, may be described by a set of basis states $|nsp_z\rangle$ (cylindrical gauge; Melrose and Parle 1983a). These states are characterised by the principle quantum number n (which determines the energy associated with motion perpendicular to the magnetic field), the component of momentum parallel to the magnetic field, p_z, and the quantum number s which determines the radial position of the guiding centre:

$$\hat{r}_0^2|nsp_z\rangle = (2s+1)(\hbar/eB)|nsp_z\rangle. \tag{1}$$

The operator for the z component of the angular momentum of the particle (assumed to be an electron) is $\hat{J}_z^e = \frac{1}{2}M\Omega(\hat{r}_1^2 - \hat{r}_0^2)$, where $\hat{r}_1^2$ is the operator for the square of the radius of gyration and $\Omega = eB/M$ is the gyrofrequency. Note that $\hat{r}_0^2$ and $\hat{r}_1^2$ are operators, and may only be replaced by the corresponding eigenvalues, $r_0^2 = (2s+1)\hbar/eB$ and $r_1^2 = 2n\hbar/eB$, when the electron is in a pure basis state $|nsp_z\rangle$. LLE fail to make this distinction: we believe it to be important in order to avoid confusion. The eigenvalues of $\hat{J}_z^e$ in relativistic theory are given by

$$\hat{J}_z^e|nsp_z\rangle = (n - s - \tfrac{1}{2})\hbar|nsp_z\rangle, \tag{2}$$

where all dependence on spin is contained in the quantum number n.

Consider emission by a single electron of a single photon of momentum $\hbar k$. Suppose that initially the state of the particle is described by a pure basis state $|nsp_z\rangle$. A relativistic calculation, which includes the magnetic field exactly (Melrose and Parle 1983b), shows that, as in the usual perturbation theory, the wavefunction of the electron after emission of the photon is described by a mixture of basis states with all values of s. If the harmonic number of emission, m (or equivalently, in non-relativistic theory, the energy of the photon, $m\hbar\Omega$), is known then the principal quantum number of the final state $n' = n - m$ is determined. The wavefunction still contains a mixture of states with different values of the radial quantum number. Suppose we measure this, and it takes the value s'. Then we argue that the wavefunction of the electron collapses to the pure basis state $|n's'p_z'\rangle$ due to the measuring process, and the angular momentum of the final state is given by

$$\hat{J}_z^e|n's'p_z'\rangle = (n' - s' - \tfrac{1}{2})\hbar|n's'p_z'\rangle.$$

Using conservation of angular momentum one argues that the angular momentum carried off by the photon is given by

$$J_z^\gamma = (n - n' - s + s')\hbar. \tag{3}$$

(Note that until one measures s' the angular momentum of the photon is not well defined. However the results of the investigation of Bell's theorem guarantee that a measurement of s' immediately fixes J_z^γ; e.g. see Clauser and Shimony 1978).

LLE use a classical expression for the vector potential of the radiation in order to calculate J_z^γ. The classical limit corresponds to the emission of a large number of photons. Suppose that we measure the eigenvalues of the operators $\hat{r}_1^2$ and $\hat{r}_0^2$ after the emission of N photons, and the quantum numbers are found to be n_f and s_f. Then the angular momentum carried off by the radiation is

$$J_z^\gamma = (n - n_f - s + s_f)\hbar. \tag{4}$$

This may be written

$$J_z^\gamma = E^\gamma/\Omega + (s_f - s)\hbar, \tag{5}$$

where $E^\gamma = (n - n_f)\hbar\Omega$ in the notation of LLE; in the non-relativistic limit E^γ is the energy carried away by the radiation. The result used by LLE is

$$J_z^\gamma = E^\gamma/\Omega. \tag{6}$$

Equations (5) and (6) are compatible only if $s = s_f$. This was also noted in Lieu (1984), who believed (6) to be correct. We argue that the calculation of (6) assumes implicitly that $s = s_f$, in that it uses a vector potential calculated on the assumption that the particle orbit (and hence s) does not change with time. We believe (6) to be incorrect: we now show that the true result is compatible with $s_f > s$ in (5).

We again consider the emission of N photons by a single electron, but calculate the expectation value of the angular momentum of the radiation field, rather than the eigenvalue. This corresponds to measuring s' for a large number of single-electron systems and averaging the results. The expectation value may be calculated using the methods described in Melrose and Parle (1983b). For emission of a single photon $\hbar \boldsymbol{k}$ by a particle initially in the pure state $|nsp_z\rangle$, one finds that if the principal quantum number is known to be n' (but the radial quantum number is undetermined) the expectation value for the angular momentum of the final state of the electron is (Parle 1985)

$$\langle \hat{J}_z^{\mathrm{e}} \rangle = (n' - s - \tfrac{1}{2} - \hbar k_\perp^2 / 2eB)\hbar, \tag{7}$$

where $\boldsymbol{k}_\perp$ is the component of $\boldsymbol{k}$ perpendicular to the magnetic field. The $k_\perp$-term in (7), which was omitted by LLE, is a recoil term which may be attributed to radiation reaction. It arises from the fact that the expectation value for s in the final state exceeds the quantum number of the initial state, cf (2). For emission of N photons whose wavevectors $\boldsymbol{k}_i$, $i = 1, \ldots, N$, are measured, the final state of the particle only depends on the total perpendicular momentum it has lost, and one replaces $\boldsymbol{k}_\perp$ in (7) with the sum of the perpendicular components of the wavevectors (Parle 1984):

$$\langle \hat{J}_z^{\mathrm{e}} \rangle = \left[n_{\mathrm{f}} - s - \tfrac{1}{2} - \frac{\hbar}{2eB} \left(\sum_{i=1}^{N} \boldsymbol{k}_{\perp i} \right)^2 \right] \hbar. \tag{8}$$

Now using $\hat{J}_z^{\mathrm{e}} = \tfrac{1}{2} M\Omega (\hat{r}_1^2 - \hat{r}_0^2)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \hat{r}_1^2 \rangle = -\frac{2}{M\Omega^2} \frac{\mathrm{d}}{\mathrm{d}t} E^\gamma, \tag{9}$$

we find that

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \hat{r}_0^2 \rangle = \frac{1}{(M\Omega)^2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{i=1}^{N} \hbar \boldsymbol{k}_{\perp i} \right)^2. \tag{10}$$

The sum on the right-hand side of (10) changes with time as successive photons are emitted (N changes). Since in writing down (10) we assume that the wavevectors of all emitted photons are measured, the sum $\Sigma_{i=1}^{N} \hbar \boldsymbol{k}_{\perp i}$ may be interpreted as the eigenvalue $\boldsymbol{P}_\perp$ of the perpendicular momentum operator for the radiation field, $\hat{\boldsymbol{P}}_\perp$ (LLE call this $\boldsymbol{P}$), after the emission of N photons. Then (10) may be written

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \hat{r}_0^2 \rangle = \frac{1}{(M\Omega)^2} \frac{\mathrm{d}}{\mathrm{d}t} P_\perp^2. \tag{11}$$

This is essentially identical to the result found by LLE using conservation of linear momentum (their equation (10)). Unfortunately LLE's notation is confusing, in that they do not distinguish between operators and the corresponding expectation values: effectively they write $P_x = P_y = 0$, but $P_\perp^2 = P_x^2 + P_y^2 > 0$, where P_x and P_y denote both operators and expectation values for the x and y components respectively of the total momentum of the radiation field.) Note that if the law of conservation of angular momentum is applied to (8) one finds that the angular momentum carried away by the radiation field is

$$J_z^\gamma = E^\gamma / \Omega + P_\perp^2 / 2M\Omega. \tag{12}$$

This differs from the result (6) used by LLE, and explains why their calculation using angular momentum provided a different result from (11).

In practice one does not measure the wavevectors of all cyclotron photons, and so the specific form of (10) is not useful. The appropriate procedure is to replace $(\Sigma_i^N \boldsymbol{k}_{\perp i})^2$

by its expectation value, measured by examining a large number of systems consisting of N cyclotron photons (this radiation expectation value is therefore different from the expectation value applying to electron operators, and we denote it by '$\langle\ \rangle_\gamma$'). The expectation value may be calculated by assuming that the wavevectors are distributed according to the cyclotron radiation pattern. We write

$$\boldsymbol{k}_{\perp i} = k_i \sin\theta_i(\cos\phi_i, \sin\phi_i, 0), \tag{13}$$

and assume that the angles ϕ_i are randomly distributed in the interval $[0, 2\pi]$ (azimuthal symmetry), while the angle θ_i are distributed according to the energy flux $j(\omega, \theta)$ used by LLE (suitably normalised). As N varies the sum $\Sigma_i^N \boldsymbol{k}_{\perp i}$ exhibits the properties of a random walk in two dimensions, and the expectation values may be calculated accordingly (Reif 1965). We find that

$$\left\langle \sum_{i=1}^{N} \boldsymbol{k}_{\perp i} \right\rangle_\gamma = 0, \tag{14}$$

as expected from azimuthal symmetry, while

$$\left\langle \left(\sum_{i=1}^{N} \boldsymbol{k}_{\perp i} \right)^2 \right\rangle_\gamma = \left\langle \sum_{i=1}^{N} k_{\perp i}^2 \right\rangle_\gamma . \tag{15}$$

The last result follows from

$$\left\langle \sum_{i=1}^{N} \sum_{j \neq i}^{N} \boldsymbol{k}_{\perp i} \cdot \boldsymbol{k}_{\perp j} \right\rangle_\gamma = \left\langle \sum_{i=1}^{N} \sum_{j \neq i}^{N} k_i k_j \sin\theta_i \sin\theta_j \cos(\phi_j - \phi_i) \right\rangle_\gamma = 0, \tag{16}$$

since the angle $\phi_j - \phi_i$ is randomly distributed. Therefore we conclude that the time evolution of r_0^2 when a large number of photons are emitted (the most likely case of interest) is described by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{r}_0^2 \rangle = \frac{1}{(M\Omega)^2} \frac{\mathrm{d}}{\mathrm{d}t} \left\langle \sum_{i=1}^{N} \hbar^2 k_{\perp i}^2 \right\rangle_\gamma > 0. \tag{17}$$

The calculation of the right-hand side of (17) is discussed in the appendix of LLE. (Note that LLE derive (11) in the body of their text, but calculate (17) in their appendix, without discussing the difference between the two equations. The discussion given here makes explicit the unstated assumptions underlying their calculation.)

Finally we comment on the work of DasGupta (1984). He reanalysed the calculation of $(\mathrm{d}/\mathrm{d}t) r_0^2$ using conservation of linear momentum, and found that $(\mathrm{d}/\mathrm{d}t) r_0^2 = 0$. His argument is that

$$\mathrm{d}P_\perp^2/\mathrm{d}t = 2P_x\, \mathrm{d}P_x/\mathrm{d}t + 2P_y\, \mathrm{d}P_y/\mathrm{d}t = 0, \tag{18}$$

since $\langle \mathrm{d}P_x/\mathrm{d}t \rangle_\gamma = \langle \mathrm{d}P_y/\mathrm{d}t \rangle_\gamma = 0$. However we believe that (18) should be written

$$(\mathrm{d}/\mathrm{d}t)\langle P_\perp^2 \rangle_\gamma = 2\langle P_x\, \mathrm{d}P_x/\mathrm{d}t \rangle_\gamma + 2\langle P_y\, \mathrm{d}P_y/\mathrm{d}t \rangle_\gamma, \tag{19}$$

and that $\langle P_x\, \mathrm{d}P_x/\mathrm{d}t \rangle_\gamma = \langle P_y\, \mathrm{d}P_y/\mathrm{d}t \rangle_\gamma \neq 0$. Essentially this error may be attributed to the confusion in the meanings of $P_\perp^2$, P_x and P_y referred to earlier. The discussion given here makes the interpretation of $P_\perp^2$ clear.

In summary, we find that the guiding centre of a particle drifts outwards from the centre of symmetry (the z axis). We note that in the classical limit a particle is described by a sum of states which typically have large n but small s. When the electron is described by a pure basis state $|nsp_z\rangle$, with $n > s$, its wavefunction is essentially confined to an annulus of mean radius $(2n\hbar/eB)^{1/2}$ from the z axis, with thickness $(6s\hbar/eB)^{1/2}$ (Parle 1984). The effect of the outward drift of the centre of gyration from the centre of symmetry (increase in s) appears as a radial spreading of the wavefunction (i.e. a thickening of the annulus), while the mean radius of the annulus diminishes as n decreases due to energy loss.

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